Lecture 13:
Tensors in paleomagnetism

- What are tensors anyway?
- Anisotropy of magnetic susceptibility (AMS)
  - How to find it
- What to do with it
- Anisotropy of magnetic remanence
What is a tensor?

“An array of numbers (or functions) that transform according to certain rules under a change of coordinates.”

“Tensors are geometric entities [that] extend the notion of scalars, vectors and matrices.”
Examples

We used a tensor of direction cosines to transform coordinate systems:

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The stress and strain tensor in structural geology

The optical indicatrix in mineralogy

AND in paleomagnetism, the relationship between the induced field and the applied field (sometimes not parallel!)
Anisotropy of Magnetic Susceptibility
In Chapter 1, we learned that \( M_I = \chi H \)

where \( \chi \) was just a number (scalar) and so \( M \) was parallel to and proportional to \( H \)

But what if the magnetic minerals in the rock are not isotropic?

\[
M_1 = \chi_{11} H_1 + \chi_{12} H_2 + \chi_{13} H_3 \\
M_2 = \chi_{21} H_1 + \chi_{22} H_2 + \chi_{23} H_3 \\
M_3 = \chi_{31} H_1 + \chi_{32} H_2 + \chi_{33} H_3
\]

or:

\[
M_i = \chi_{ij} H_j
\]
\[ M_i = \chi_{ij} H_j \]

For convenience (later on), we can remap this to:

\[
\begin{align*}
  s_1 &= \chi_{11} \\
  s_2 &= \chi_{22} \\
  s_3 &= \chi_{33} \\
  s_4 &= \chi_{12} = \chi_{21} \\
  s_5 &= \chi_{23} = \chi_{32} \\
  s_6 &= \chi_{13} = \chi_{31}
\end{align*}
\]
Applications of Anisotropy of magnetic susceptibility

- Related to distribution of magnetic minerals (easy axes) in rocks
- Can use it to understand rock fabrics
  - Sedimentary processes
  - Igneous processes
  - Metamorphic processes
need to measure in at least 6 positions to determine AMS tensor

Measured values are $K_i$

- $K_1 = s_1$
- $K_2 = s_2$
- $K_3 = s_3$

**BUT**

- $K_4 = \frac{1}{2}(s_1 + s_2) + s_4$
- $K_5 = \frac{1}{2}(s_2 + s_3) + s_5$
- $K_6 = \frac{1}{2}(s_1 + s_3) + s_6$
\[ K_1 = s_1 \quad K_4 = \frac{1}{2}(s_1 + s_2) + s_4 \]
\[ K_2 = s_2 \quad K_5 = \frac{1}{2}(s_2 + s_3) + s_5 \]
\[ K_3 = s_3 \quad K_6 = \frac{1}{2}(s_1 + s_3) + s_6 \]

can be re-written as: \( K_i = A_{ij} s_j \) where \( A_{ij} \) is:

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
.5 & .5 & 0 & 1 & 0 & 0 & 0 \\
0 & .5 & .5 & 0 & 1 & 0 & 0 \\
.5 & 0 & .5 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
So how do we get the $s_i$ from the measurements $K_i$?

We need tricks from linear algebra:

$$
\bar{s} = (A^T A)^{-1} A^T K \quad \text{or} \quad \bar{s}_i = B_{ij} K_j
$$

where $B$ is:

$$
B = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-0.5 & -0.5 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -0.5 & -0.5 & 0 & 1 & 0 & 0 & 0 \\
-0.5 & 0 & -0.5 & 0 & 0 & 1 & 0 & 0 \\
-0.5 & 0 & -0.5 & 0 & 0 & 1 & 0 & 0 \\
-0.5 & 0 & -0.5 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
$$
In general, \( M \) is not parallel to \( H \) because coefficients

\[
\chi_{ij} \neq 0 \text{ when } i \neq j
\]
But there exists a coordinate system where $M$ IS parallel to $H$. Axes are called “eigenvectors”: $V_i$

so when $H \parallel V_1$, $M \parallel V_1$

and the magnitude corresponds to: $M = \chi_{max} H$
see Appendix A.3.5.4 for how to find eigenvectors and eigenvalues

where $\tau_i$ are the eigenvalues

$\tau_1 > \tau_2 > \tau_3$
What is AMS good for?

- Is a particular fabric axis parallel to some direction? e.g., is the minimum axes parallel to the vertical as expected for sediments?
- Are two sets of axes distinct? e.g., has strain rotated the rock fabric?
- What is the shape of the ellipsoid? Is it “pancake” (as exp. for sediments?) or “hot-dog” (as exp. for deformed fabrics?)

These all require statistical test! – poor statistics with just 6 measurements
Statistics for AMS data

• Every measurement has some uncertainty:
  \[ K_i = A_{ij} \bar{s}_j + \delta_i \]

• Remember we get average tensor elements::
  \[ \bar{s}_i = B_{ij} K_j \]

• Now we could calculate what measurements we “should” have gotten, assuming that the average tensor was correct:
  \[ \bar{K}_i = A_{ij} \bar{s}_j \]

• and calculate the deviations (residuals) between “average” and actual measurements:
  \[ \delta_i = K_i - \bar{K}_i \]
The fun begins

• A measure of the uncertainty is the sum of squared residuals:

\[ S_o = \sum \delta_i^2 \]

• From this we get the estimated variance:

\[ \sigma^2 = S_o / n_f \]

• But what is “degrees of freedom”, nf? It is the number of measurements less the number of things you are estimating

\[ N_{meas} - 6 \]

• So we need more than six measurements to do any cool statistics
Measurement of AMS in practice

- Lots of different protocols (7, 12, 15, “many” measurement protocols)

- In Appendix D.1 you will find descriptions of two commonly used ones: 15 and the “spinning” protocols

- Each protocol has its own “design matrix”, $B$ from which all data reduction flows
Parametric significance tests

- Analogous to Fisher statistics – known as Hext statistics
- Allows calculation of uncertainties in eigenvector directions
- Allows tests for shape (are the eigenvalues significantly different so what shape do we have?)
Uncertainty ellipses of eigenvectors: the parametric approach of Hext

\[ \epsilon_{12} = \tan^{-1} \left[ \frac{f \sigma}{2(\tau_1 - \tau_2)} \right] \]
\[ \epsilon_{23} = \tan^{-1} \left[ \frac{f \sigma}{2(\tau_2 - \tau_3)} \right] \]
\[ \epsilon_{13} = \tan^{-1} \left[ \frac{f \sigma}{2(\tau_1 - \tau_3)} \right] \]
\[ \epsilon_{21} = \epsilon_{12} \]
\[ \epsilon_{32} = \epsilon_{23} \]
\[ \epsilon_{31} = \epsilon_{13} , \]

where
\[ f = \sqrt{2 \left( F_{(2,n_f);(1-p)} \right)} \]

and this F is another critical value for significance at a given level of confidence (e.g., \( p = .05 \) for 95%) look up F on an “F-table”
Shapes - the Hext way:
but first some definitions

sphere

\[ \tau_1 \sim \tau_2 \sim \tau_3 \]

prolate

\[ \tau_1 > \tau_2 \sim \tau_3 \]

oblate

\[ \tau_1 \sim \tau_2 > \tau_3 \]

triaxial

\[ \tau_1 > \tau_2 > \tau_3 \]
F statistics for shapes

but first, bulk susceptibility:

\[ \chi_b = \left( \bar{s}_1 + \bar{s}_2 + \bar{s}_3 \right)/3 \]

Are your data spherical? Must have F less than this:

\[ F = 0.4 \left( \tau_1^2 + \tau_2^2 + \tau_3^2 - 3\chi_b^2 \right)/\sigma^2 \]

Are they oblate? \[ F_{23} = 0.5((\tau_2 - \tau_3)/\sigma)^2 \]

Are they prolate? \[ F_{12} = 0.5((\tau_1 - \tau_2)/\sigma)^2 \]
But when do Hext statistics apply?

- When errors are small
- When they have zero mean
- AND when they are normally distributed
- oops - only for measurement data - rarely for data from multiple specimens
- maybe you should bootstrap?
Bootstrapping for AMS
works just like for directions: see Chapter 12

Data

bootstrapped eigenvectors

bootstrapped eigenvalues

Eigenvalues

0.20 0.25 0.30 0.35 0.40 0.45 0.50

0.0 0.2 0.4 0.6 0.8 1.0
What is AMS good for?

- Is a particular fabric axis parallel to some direction? e.g., is the minimum axes parallel to the vertical as expected for sediments?

- Are two sets of axes distinct? e.g., has strain rotated the rock fabric?

- What is the shape of the ellipsoid? Is it “pancake” (as exp. for sediments?) or “hot-dog” (as exp. for deformed fabrics?)
Do these data have a vertical minimum axis?
Some comments on AMS statistics

There are many ways of repackaging the eigenvalues into anisotropy statistics (see Table 13.1)

Typical examples boil the eigenvalues into an anisotropy factor (P or P’) and one that reflects shape (e.g., shape factor, T)

this allows only “prolate” and “oblate” – what about spherical and triaxial? you lose info

also none incorporate any uncertainty estimate
Anisotropy of magnetic remanence

Different than susceptibility because can measure three components at a time - not just one.

So degrees of freedom are calculated differently (nf= 3\times N_{meas}-6 instead of N_{meas}-6)

Useful for:

- correcting absolute paleointensity data for anisotropy of TRM
- correcting sedimentary data for inclination error
Remember from Chapter 10 that TRM can be anisotropic:
Analogous to induced magnetization and susceptibility:

\[ M_I = \chi H \]

Remanent vectors can be related to the applied field by:

\[ M_R = \chi^R H \]

Where \( \chi^R \) is the anisotropy of remanence tensor.

Knowing that, we could get:

\[ H_{anc} = M \cdot \chi_R^{-1} \]

(see Chapter 13 for tricky bits regarding normalization)
TRM anisotropy in practice

• Give specimens total TRM in at least 3 (usually 6 or 9 for uncertainties) directions.

• Calculate remanence anisotropy tensor using the design matrix for your protocol.

• Because each additional heating step can produce additional alteration, some substitute ARM anisotropy for TRM anisotropy

• With ARMs, you have to AF demagnetize completely between each ARM acquisition step
Remember inclination “error” (Chapter 7)?

DRM anisotropy may help us to correct for this
As before: \[ \mathbf{M}_{DRM} = \chi^{DRM} \mathbf{H} \]

But in the book, we use terminology of Jackson et al. (1991) or: \[ \mathbf{M}_d = k_d \mathbf{H} \]

DRM anisotropy tensor has eigenvalues: \[ \kappa_{d1} > \kappa_{d2} > \kappa_{d3} \]

flattening factor, f, gotten by: \[ f = \frac{\kappa_{d3}}{\kappa_{d1}} \]

so all we need is the DRM anisotropy tensor!

But, this is not so easy.....
How to get DRM anisotropy tensor:

• Redeposition? NOT
• Measure proxy? ARM?
  • needs to reflect same grains as those carrying the DRM
  • needs to take into account “particle anisotropy”
• what about the effect of flocculation?
• for details see Chapter 13 and Appendix D.3
Assignment

Problem 13.3