Lecture 13:

Tensors in paleomagnetism

- What are tensors anyway?
- anisotropy of magnetic susceptibility (AMS)
  - how to find it
  - what to do with it
- anisotropy of magnetic remanence

What is a tensor?

- "An array of numbers (or functions) that transform according to certain rules under a change of coordinates."
- "Tensors are geometric entities [that] extend the notion of scalars, vectors and matrices."

Examples

- We used a tensor of direction cosines to transform coordinate systems:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

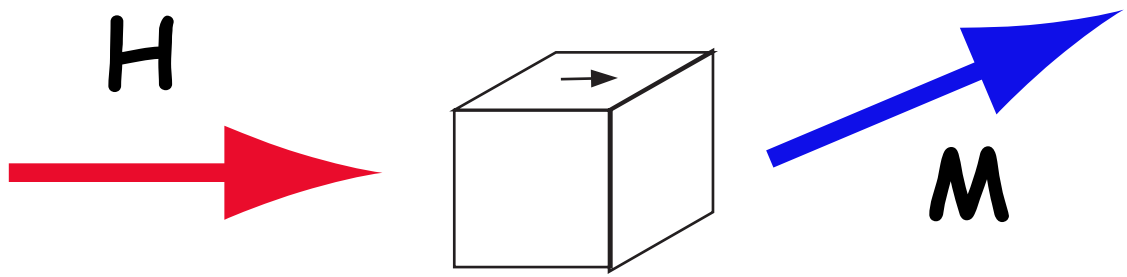
- The stress and strain tensor in structural geology
- The optical indicatrix in mineralogy
- AND in paleomagnetism, the relationship between the induced field and the applied field (sometimes not parallel!)

Anisotropy of Magnetic Susceptibility

In Chapter 1, we learned that $M_I = \chi H$

where χ was just a number (scalar) and so M was parallel to and proportional to H

But what if the magnetic minerals in the rock are not isotropic?



$$M_1 = \chi_{11} H_1 + \chi_{12} H_2 + \chi_{13} H_3$$
$$M_2 = \chi_{21} H_1 + \chi_{22} H_2 + \chi_{23} H_3$$
$$M_3 = \chi_{31} H_1 + \chi_{32} H_2 + \chi_{33} H_3$$

or: $M_i = \chi_{ij} H_j$

$$M_i = \chi_{ij} H_j$$

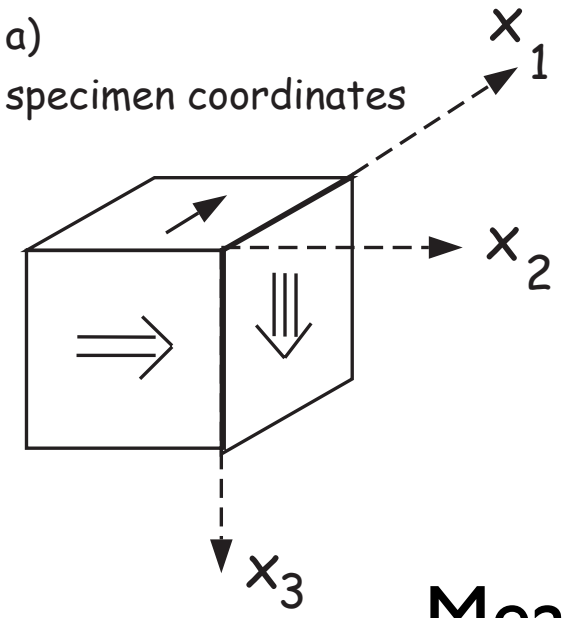
For convenience (later on), we can remap this to:

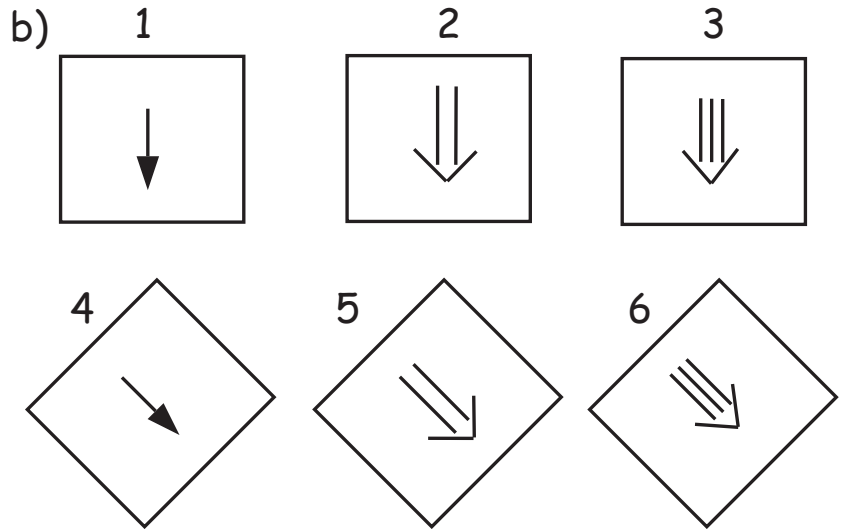
$$s_1 = \chi_{11}$$
$$s_2 = \chi_{22}$$
$$s_3 = \chi_{33}$$
$$s_4 = \chi_{12} = \chi_{21}$$
$$s_5 = \chi_{23} = \chi_{32}$$
$$s_6 = \chi_{13} = \chi_{31}$$

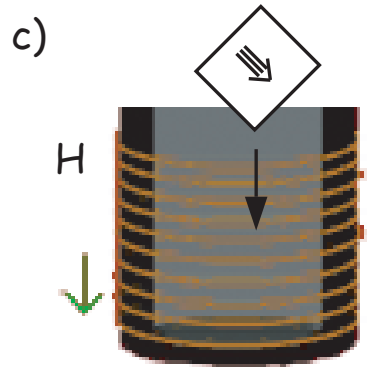
Applications of Anisotropy of magnetic susceptibility

- Related to distribution of magnetic minerals (easy axes) in rocks
- Can use it to understand rock fabrics
  - Sedimentary processes
  - Igneous processes
  - Metamorphic processes

need to measure in at least 6 positions to determine AMS tensor







Measured values are K_i

$$K_1 = s_1$$

$$K_2 = s_2$$

$$K_3 = s_3$$

BUT

$$K_4 = \frac{1}{2}(s_1 + s_2) + s_4$$

$$K_5 = \frac{1}{2}(s_2 + s_3) + s_5$$

$$K_6 = \frac{1}{2}(s_1 + s_3) + s_6$$

$$K_1 = s_1 \qquad K_4 = \frac{1}{2}(s_1 + s_2) + s_4$$
$$K_2 = s_2 \qquad K_5 = \frac{1}{2}(s_2 + s_3) + s_5$$
$$K_3 = s_3 \qquad K_6 = \frac{1}{2}(s_1 + s_3) + s_6$$

can be re-written as: $K_i = A_{ij} s_j$ where A_{ij} is:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ .5 & .5 & 0 & 1 & 0 & 0 \\ 0 & .5 & .5 & 0 & 1 & 0 \\ .5 & 0 & .5 & 0 & 0 & 1 \end{pmatrix}$$

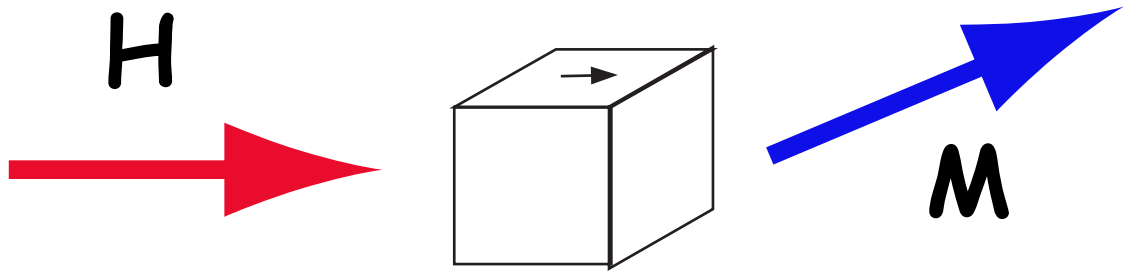
So how do we get the s_i from the measurements K_i?

need tricks from linear algebra:

$$\bar{s} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{K} \quad \text{or} \quad \bar{s}_i = B_{ij}K_j$$

where B is:

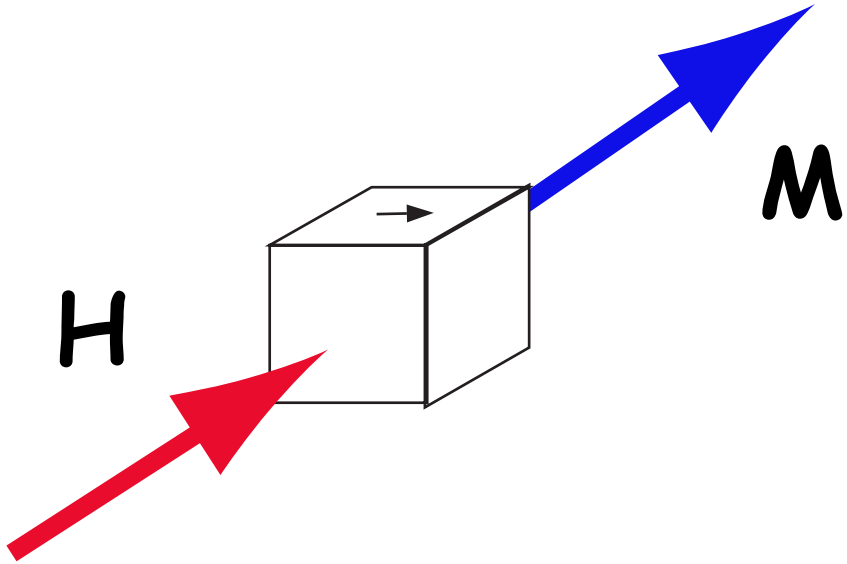
$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -.5 & -.5 & 0 & 1 & 0 & 0 \\ 0 & -.5 & -.5 & 0 & 1 & 0 \\ -.5 & 0 & -.5 & 0 & 0 & 1 \end{pmatrix}$$

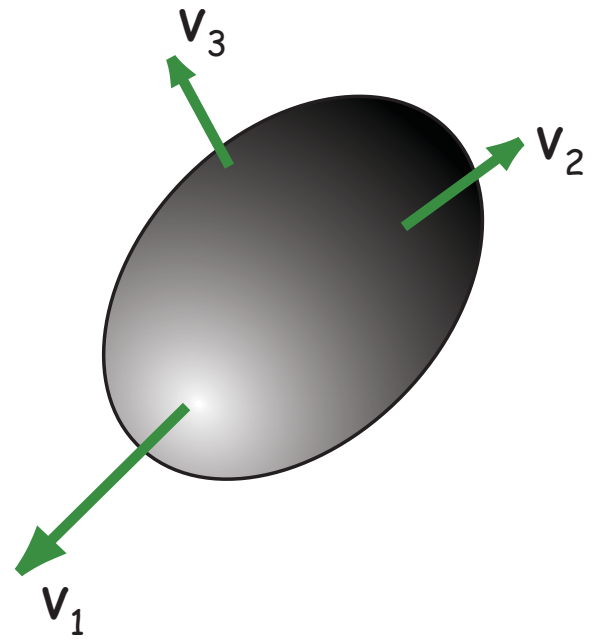


$$M_1 = \chi_{11} H_1 + \chi_{12} H_2 + \chi_{13} H_3$$
$$M_2 = \chi_{21} H_1 + \chi_{22} H_2 + \chi_{23} H_3$$
$$M_3 = \chi_{31} H_1 + \chi_{32} H_2 + \chi_{33} H_3$$

In general, M is not parallel to H because coefficients

$$\chi_{ij} \neq 0 \text{ when } i \neq j$$

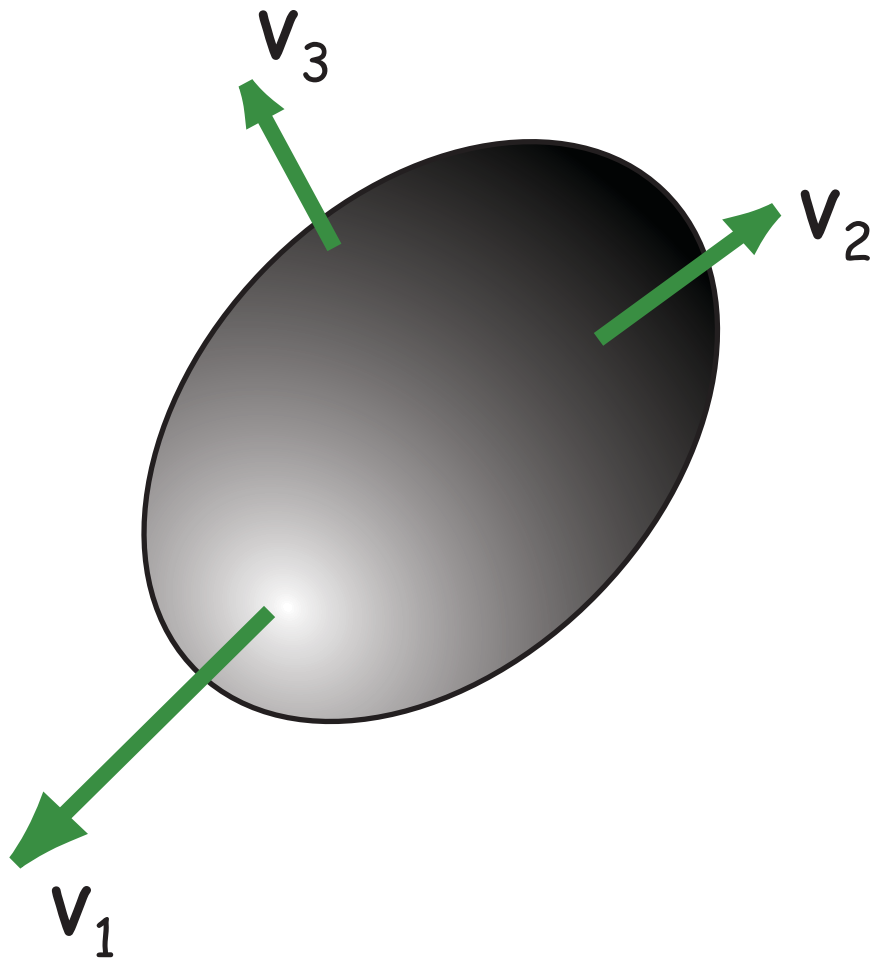




But there exists a coordinate system where M IS parallel to H.
Axes are called "eigenvectors": $\mathbf{V_i}$

$$\text{so when } \mathbf{H} \parallel \mathbf{V_1}, \mathbf{M} \parallel \mathbf{V_1}$$

$$\text{and the magnitude corresponds to: } M = \chi_{max} H$$



see Appendix A.3.5.4 for how to find eigenvectors and eigenvalues

$$\chi_{max} \propto \tau_1$$
$$\chi_{int} \propto \tau_2$$
$$\chi_{min} \propto \tau_3$$

where τ_i are the eigenvalues

$$\tau_1 > \tau_2 > \tau_3$$

What is AMS good for?

- Is a particular fabric axis parallel to some direction? e.g., is the minimum axes parallel to the vertical as expected for sediments?
- Are two sets of axes distinct? e.g., has strain rotated the rock fabric?
- What is the shape of the ellipsoid? Is it "pancake" (as exp. for sediments?) or "hot-dog" (as exp. for deformed fabrics?)

These all require statistical test! - poor statistics with just 6 measurements

Statistics for AMS data

- Every measurement has some uncertainty:

$$K_i = A_{ij}\bar{s}_j + \delta_i$$

- Remember we get average tensor elements::

$$\bar{s}_i = B_{ij}K_j$$

- Now we could calculate what measurements we "should" have gotten, assuming that the average tensor was correct:

$$\bar{K}_i = A_{ij}\bar{s}_j$$

- and calculate the deviations (residuals) between "average" and actual measurements:

$$\delta_i = K_i - \bar{K}_i$$

The fun begins

- A measure of the uncertainty is the sum of squared residuals:

$$S_o = \sum_i \delta_i^2$$

- From this we get the estimated variance:

$$\sigma^2 = S_o / n_f$$

- But what is "degrees of freedom", nf? It is the number of measurements less the number of things you are estimating

$$N_{meas} - 6$$

- So we need more than six measurements to do any cool statistics

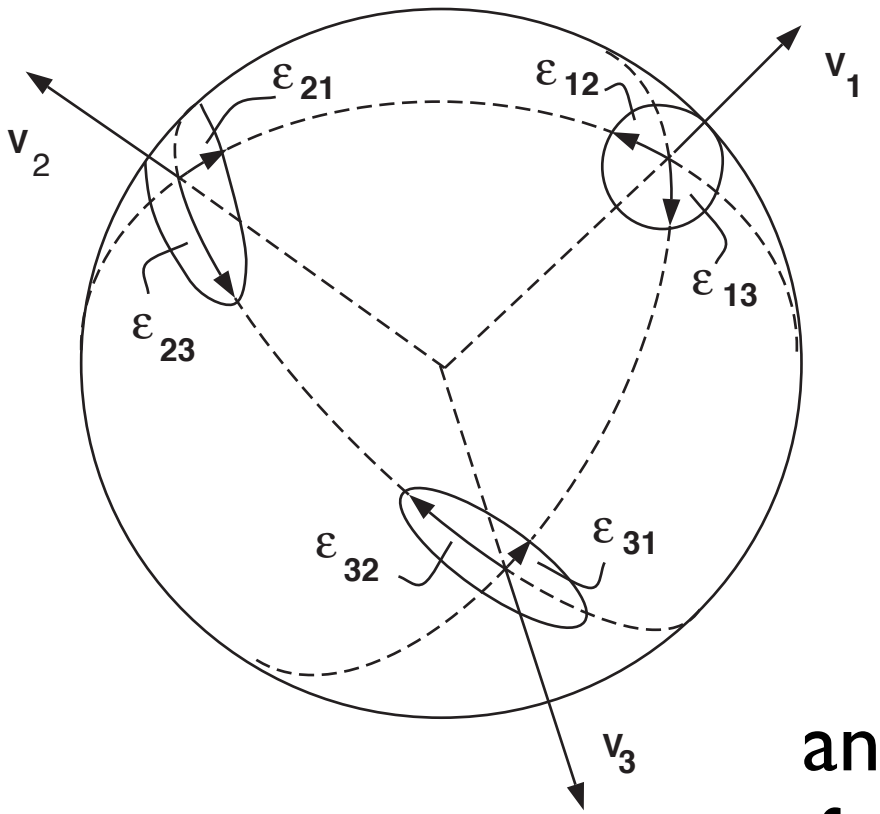
Measurement of AMS in practice

- Lots of different protocols (7, 12, 15, "many" measurement protocols)
- In Appendix D.1 you will find descriptions of two commonly used ones: 15 and the "spinning" protocols
- Each protocol has its own "design matrix", B from which all data reduction flows

Parametric significance tests

- Analogous to Fisher statistics - known as Hext statistics
- Allows calculation of uncertainties in eigenvector directions
- Allows tests for shape (are the eigenvalues significantly different so what shape do we have?)

Uncertainty ellipses of eigenvectors: the parametric approach of Hext

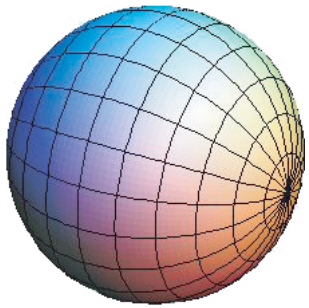


$$\epsilon_{12} = \tan^{-1}[f\sigma/2(\tau_1 - \tau_2)]$$
$$\epsilon_{23} = \tan^{-1}[f\sigma/2(\tau_2 - \tau_3)]$$
$$\epsilon_{13} = \tan^{-1}[f\sigma/2(\tau_1 - \tau_3)]$$
$$\epsilon_{21} = \epsilon_{12}$$
$$\epsilon_{32} = \epsilon_{23}$$
$$\epsilon_{31} = \epsilon_{13},$$

where $f = \sqrt{2(F_{(2,n_f);(1-p)})}$

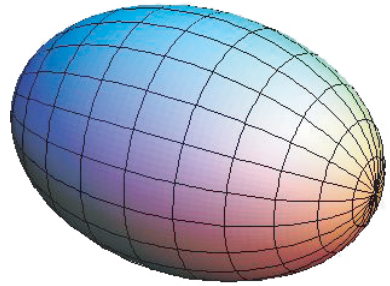
and this F is another critical value for significance at a given level of confidence (e.g., p = .05 for 95%)look up F on an "F-table"

Shapes - the Hext way: but first some definitions



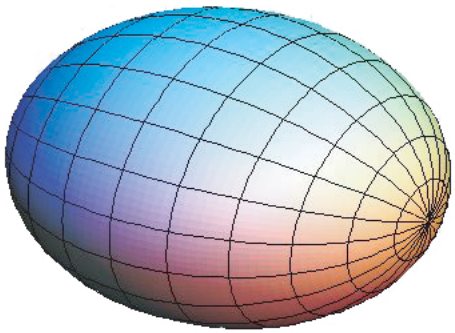
sphere

$\tau_1 \simeq \tau_2 \simeq \tau_3$



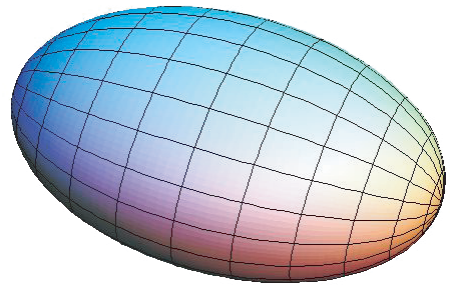
prolate

$\tau_1 > \tau_2 \simeq \tau_3$



oblate

$\tau_1 \simeq \tau_2 > \tau_3$



triaxial

$\tau_1 > \tau_2 > \tau_3$

F statistics for shapes

but first, bulk susceptibility:

$$\chi_b = (\bar{s}_1 + \bar{s}_2 + \bar{s}_3)/3$$

Are your data spherical? Must have F less than this:

$$F = 0.4\left(\tau_1^2 + \tau_2^2 + \tau_3^2 - 3\chi_b^2\right)/\sigma^2$$

Are they oblate? $F_{23} = 0.5((\tau_2 - \tau_3)/\sigma)^2$

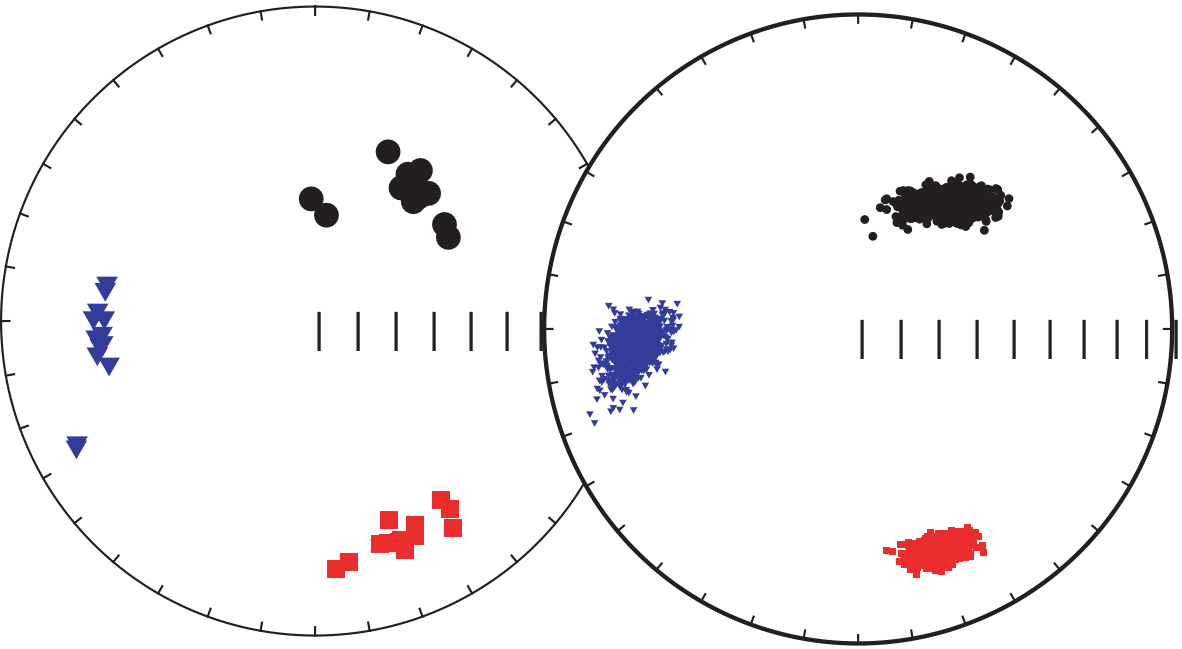
Are they prolate? $F_{12} = 0.5((\tau_1 - \tau_2)/\sigma)^2$

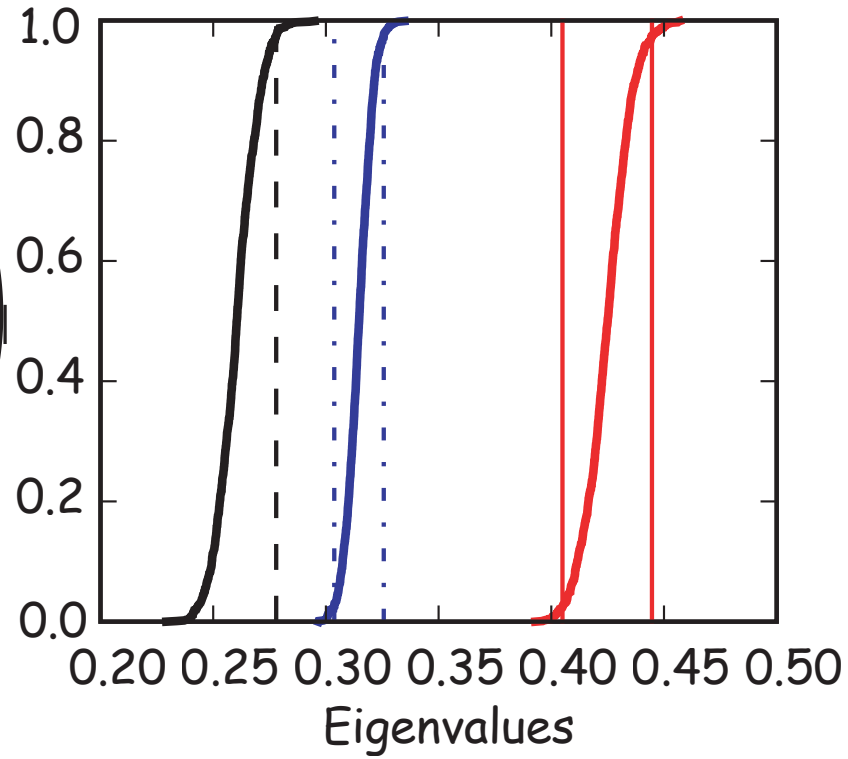
But when do Hext statistics apply?

- When errors are small
- When they have zero mean
- AND when they are normally distributed
- oops - only for measurement data - rarely for data from multiple specimens
- maybe you should bootstrap?

Bootstrapping for AMS

works just like for directions: see Chapter 12





Data

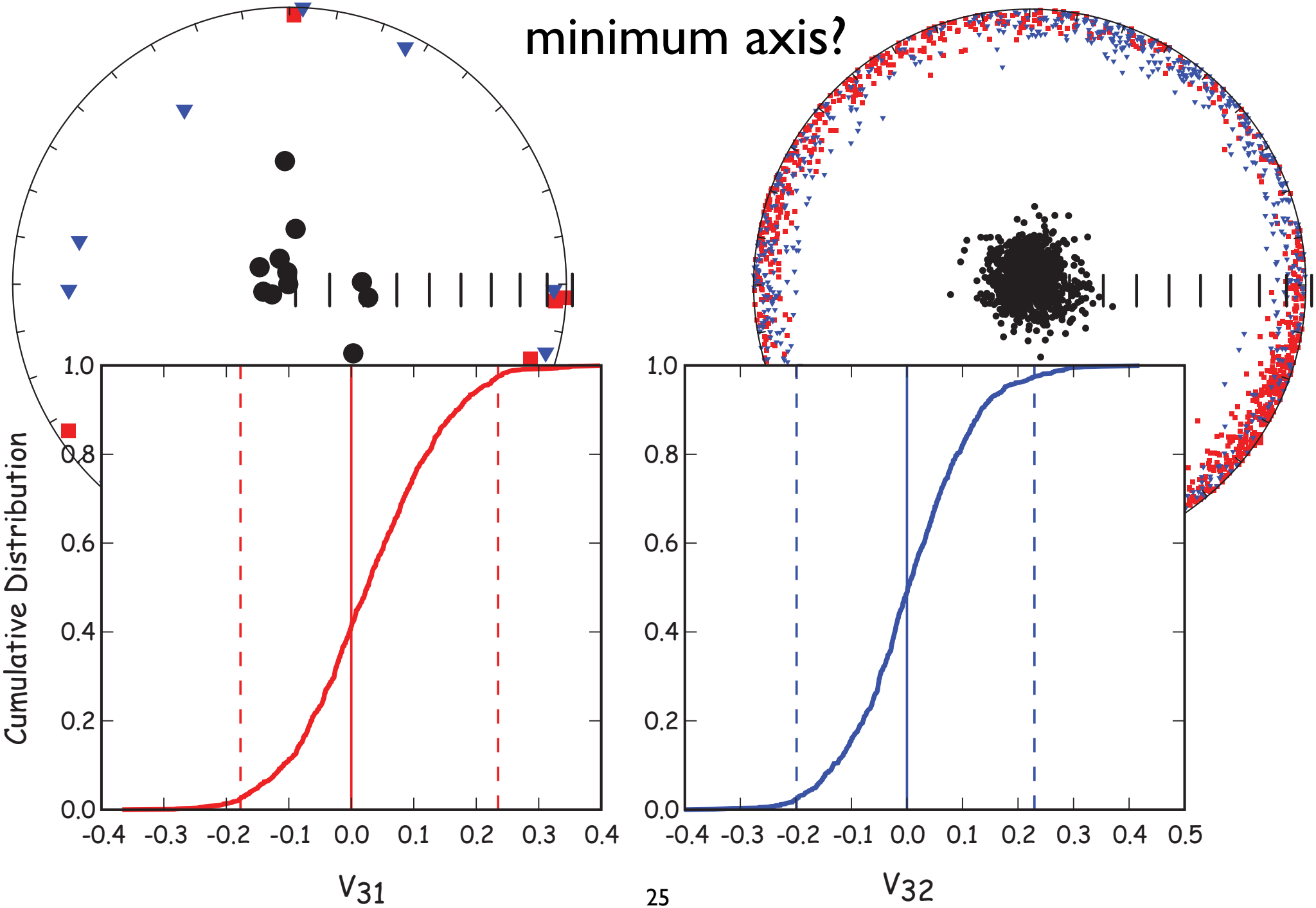
bootstrapped eigenvectors

bootstrapped eigenvalues

What is AMS good for?

- Is a particular fabric axis parallel to some direction? e.g., is the minimum axes parallel to the vertical as expected for sediments?
- Are two sets of axes distinct? e.g., has strain rotated the rock fabric?
- What is the shape of the ellipsoid? Is it "pancake" (as exp. for sediments?) or "hot-dog" (as exp. for deformed fabrics?)

Do these data have a vertical minimum axis?



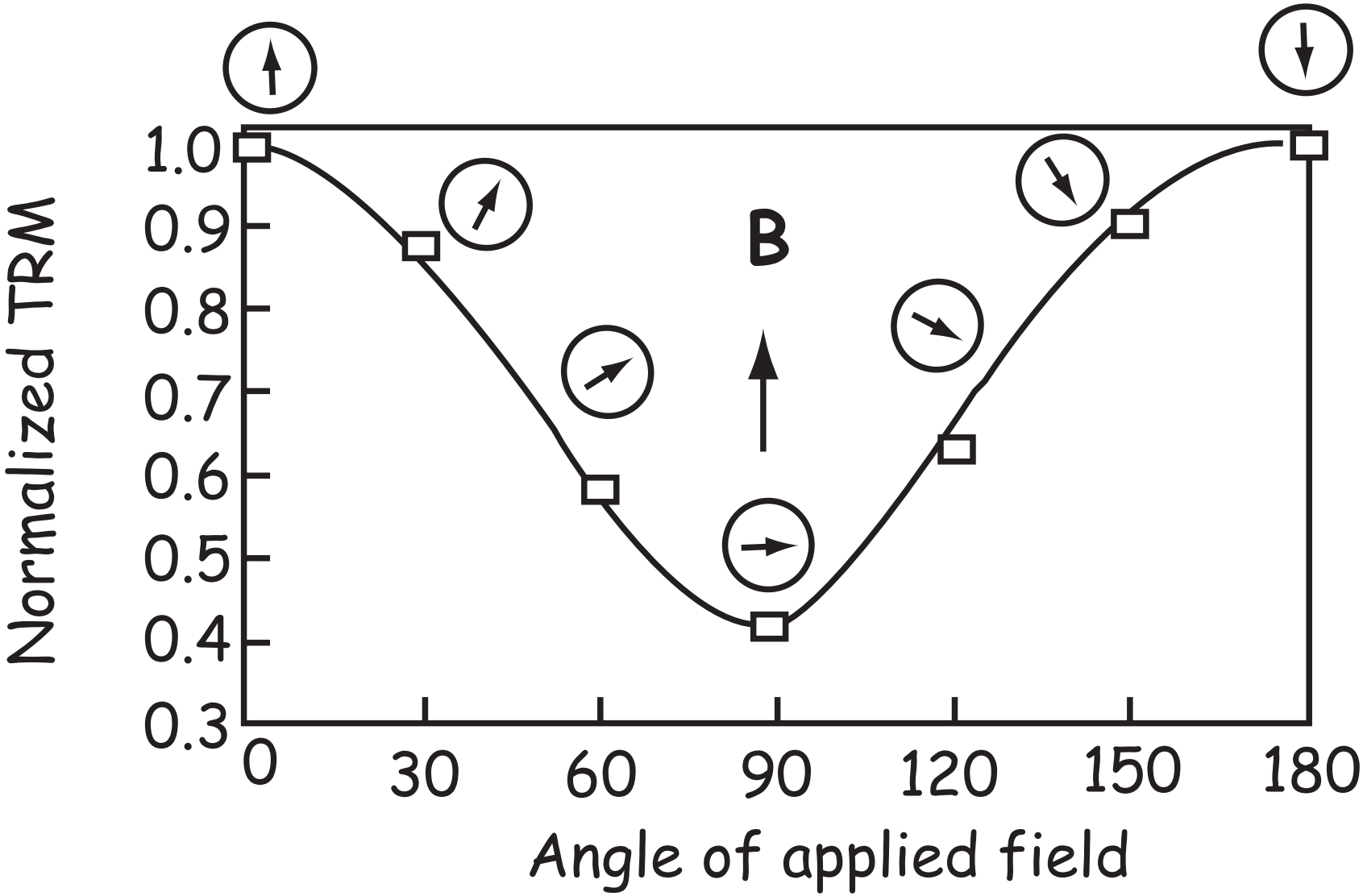
Some comments on AMS statistics

- There are many ways of repackaging the eigenvalues into anisotropy statistics (see Table 13.1)
- Typical examples boil the eigenvalues into an anisotropy factor (P or P′) and one that reflects shape (e.g., shape factor, T)
- this allows only "prolate" and "oblate" - what about spherical and triaxial? you lose info
- also none incorporate any uncertainty estimate

Anisotropy of magnetic remanence

- Different than susceptibility because can measure three components at a time - not just one.
- So degrees of freedom are calculated differently (nf= 3xNmeas-6 instead of Nmeas-6)
- Useful for:
  - correcting absolute paleointensity data for anisotropy of TRM
  - correcting sedimentary data for inclination error

Remember from Chapter 10
that TRM can be anisotropic:



Analogous to induced magnetization and susceptibility:

$$\mathbf{M_I} = \chi \mathbf{H}$$

Remanent vectors can be related to the applied field by:

$$\mathbf{M_R} = \chi^{R} \mathbf{H}$$

Where χ^{R} is the anisotropy of remanence tensor

Knowing that, we could get:

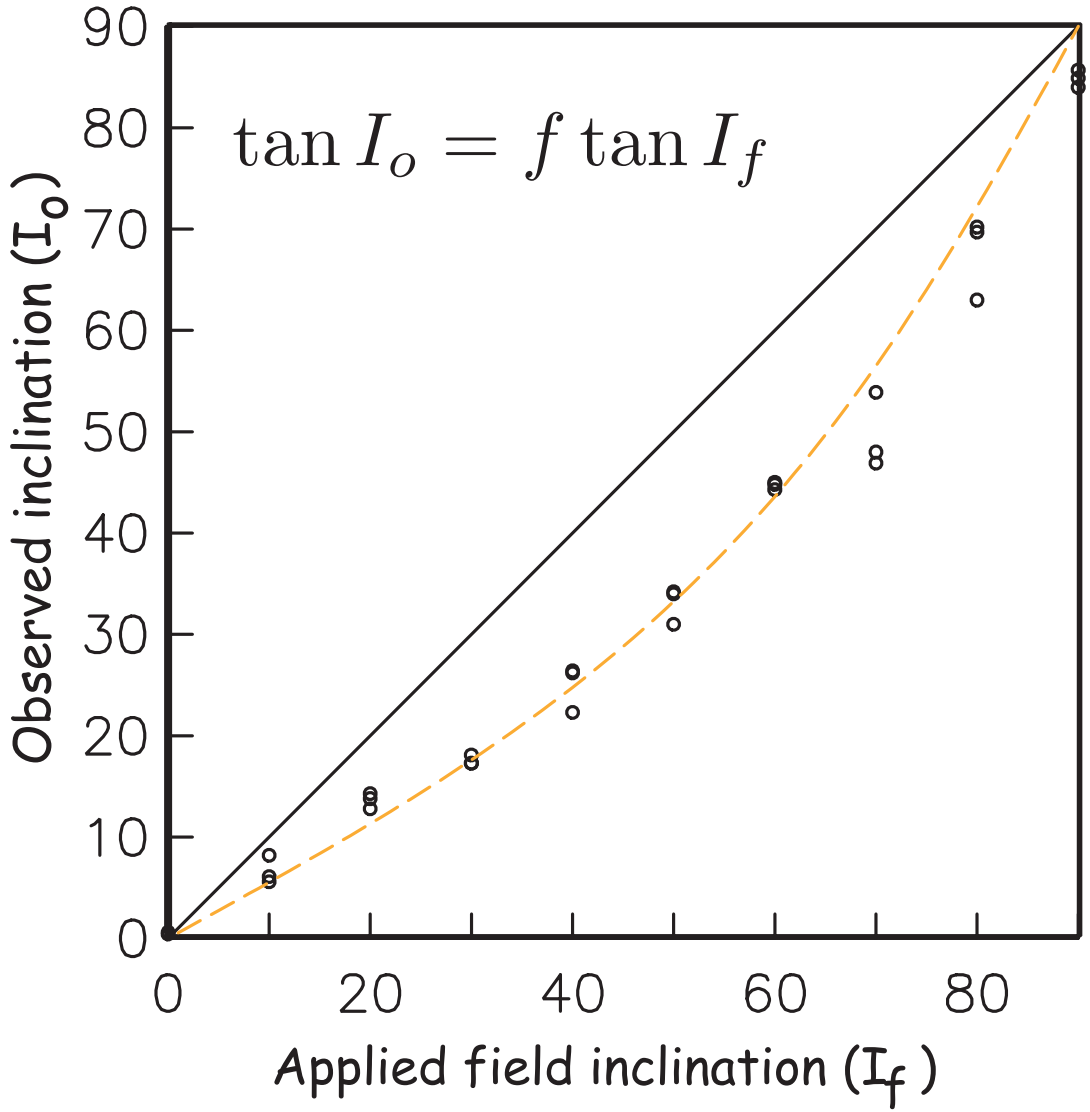
$$\mathbf{H}_{anc} = \mathbf{M} \cdot {\chi_R}^{-1}$$

(see Chapter 13 for tricky bits regarding normalization)

TRM anisotropy in practice

- Give specimens total TRM in at least 3 (usually 6 or 9 for uncertainties) directions.
- Calculate remanence anisotropy tensor using the design matrix for your protocol.
- Because each additional heating step can produce additional alteration, some substitute ARM anisotropy for TRM anisotropy
- With ARMs, you have to AF demagnetize completely between each ARM acquisition step

Remember inclination "error" (Chap ter 7)?



DRM anisotropy may help us to correct for this

As before: $\mathbf{M}_{DRM} = \chi^{DRM}\mathbf{H}$

But in the book, we use terminology of Jackson et al. (1991) or: $\mathbf{M}_d = \mathbf{k_d}\mathbf{H}$

DRM anisotropy tensor has eigenvalues:

$$\kappa_{d1} > \kappa_{d2} > \kappa_{d3}$$

flattening factor, f, gotten by: $f = \frac{\kappa_{d3}}{\kappa_{d1}}$

so all we need is the DRM anisotropy tensor!

But, this is not so easy.....

How to get DRM anisotropy tensor:

- Redeposition? NOT
- Measure proxy? ARM?
  - needs to reflect same grains as those carrying the DRM
  - needs to take into account "particle anisotropy"
  - what about the effect of flocculation?
  - for details see Chapter 13 and Appendix D.3

Assignment

- Problem 13.3